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# Generalized Maxwell state and $H$ theorem for computing fluid flows using the lattice Boltzmann method

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Generalized Maxwell distribution function is derived analytically for the lattice Boltzmann (LB) method. All the previously introduced equilibria for LB are found as special cases of the generalized Maxwellian. The generalized Maxwellian is used to derive a different class of multiple relaxation-time LB models and prove the $H$ theorem for them.

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A branch of kinetic theory—the lattice Boltzmann (LB) method—has recently met with a remarkable success as a powerful alternative for solving the hydrodynamic Navier-Stokes equations, with applications ranging from large Reynolds number flows to flows at a micron scale, porous media, and multiphase flows, see, e.g., [1–3] and references therein. The LB method solves a fully discrete kinetic equation for populations $f_\alpha(\boldsymbol{x},t)$, designed in a way that it reproduces the Navier-Stokes equations in the hydrodynamic limit in $D$ dimensions. Populations correspond to discrete velocities $\boldsymbol{v}_\alpha$ for $\alpha=0,1,\ldots,Q-1$, which fit into a regular spatial lattice with the nodes $\boldsymbol{x}$. This enables a simple and highly efficient algorithm based on (a) nodal relaxation and (b) streaming along the links of the regular spatial lattice. On the other hand, numerical stability of the LB method remains a critical issue [2]. Recalling the role played by the Boltzmann $H$ theorem in enforcing macroscopic evolutionary constraints (the second law of thermodynamics), pertinent entropy functions have been proposed [4–8]. The full connection of LB to kinetic theory was established by the discrete-velocity analog of the Maxwellian [see Eq. (2) below].

Admittedly, however, other heuristic methods were proposed recently to enhance stability of LB. The rationale behind one of them, the multiple relaxation time (MRT) [9–11], is sound. Since the incompressible flow is the only concern, the bulk viscosity arising in the quasicompressible LB scheme can be viewed as a free parameter and tuned in order to enhance stability. However, in spite of popularity of the MRT method, to date, it cannot be considered as a consistent kinetic theory but rather a numerical trick where tuning of parameters is based on experience rather than on physics.

In this paper, we present a different consideration of the LB models and derive a crucial result: the closed-form generalized equilibrium [see Eq. (3) below]. The generalized equilibrium is the analog of the anisotropic Gaussian and is a long-needed relevant distribution in the LB method. This finding further allows us to introduce an innovative class of entropy-based MRT LB models which enjoy both the $H$ theorem and the additional free-tunable parameter for controlling the bulk viscosity, where the range is dictated by the entropy.

For the sake of presentation and without any loss of generality, we consider the popular nine-velocity model, the so-called D2Q9 lattice, of which the discrete velocities are $\boldsymbol{v}_0=(0,0)$ and $\boldsymbol{v}_i=(\pm c,0)$ and $(0,\pm c)$ for $i=1-4$, and $\boldsymbol{v}_i=(\pm c,\pm c)$ for $i=5-8$ [12] where $c$ is the lattice spacing. Recall that the D2Q9 lattice derives from the three-point Gauss-Hermite formula [13] with the following weights of the quadrature $w(-1)=1/6$, $w(0)=2/3$, and $w(+1)=1/6$. Let us arrange in the list $v_x$ all the components of the lattice velocities along the $x$ axis and similarly in the list $v_y$. Analogously let us arrange in the list $f$ all the populations $f_\alpha$. Algebraic operations for the lists are always assumed component wise. The sum of all the elements of the list $p$ is denoted by $\langle p\rangle=\sum_{i=0}^{Q-1}p_i$. The dimensionless density $\rho$, the flow velocity $\boldsymbol{u}$, and the second-order moment (pressure tensor) $\boldsymbol{\Pi}$ are defined by $\rho=\langle f\rangle$, $\rho u_i=\langle v_i f\rangle$, and $\rho\Pi_{ij}=\langle v_i v_j f\rangle$, respectively.

On the lattice under consideration, the convex entropy function ($H$ function) is defined as [5]

$$H(f)=\langle f\ln(f/W)\rangle, \tag{1}$$

where $W=w(v_x)w(v_y)$. The $H$-function minimization problem is considered in the sequel. It is well known [5] that the equilibrium population list $f_M$ is defined as the solution of the minimization problem $f_M=\min_{f\in \mathbf{P}_M}H(f)$, where $\mathbf{P}_M$ is the set of functions such that $\mathbf{P}_M=\{f>0:\langle f\rangle=\rho,\ \langle \boldsymbol{v}f\rangle=\rho\boldsymbol{u}\}$. In other words, minimization of the $H$ function Eq. (1) under the constraints of mass and momentum conservation yields [6]

$$f_M=\rho\prod_{\alpha=x,y}w(v_\alpha)[2-\varphi(u_\alpha/c)]\left(\frac{2(u_\alpha/c)+\varphi(u_\alpha/c)}{1-(u_\alpha/c)}\right)^{v_\alpha/c}, \tag{2}$$

where $\varphi(z)=\sqrt{3z^2+1}$. A remarkable feature of equilibrium (2) which it shares with the ordinary Maxwellian is that it is a product of one-dimensional equilibria. In order to ensure the positivity of $f_M$, the low Mach number limit must be considered, i.e., $|u_\alpha|<c$.

In this paper, we derive a different constrained equilibrium, or quasiequilibrium [14], by requiring, in addition, that the diagonal components of the pressure tensor $\boldsymbol{\Pi}$ have some prescribed values. Hence let us introduce a different minimization problem. The quasiequilibrium population list $f_G$ is

 

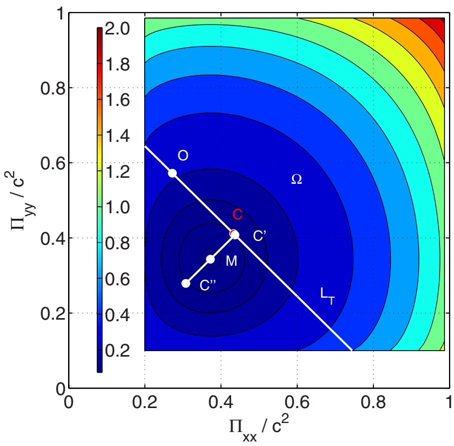


FIG. 1. (Color online) Contour plot of the entropy $H_G$ [Eq. (4)] at $\rho=1$, $u_x=-0.2$, and $u_y=0.1$ ($c=1$). Rectangular domain is the positivity domain $\Omega$. $M$ is the image of Maxwellian (2). $O$ is the image of a generic nonequilibrium state while $C$ is the image of the constrained equilibrium (7) (minimum of $H_G$ on the line $L_T$). $C'$ is the low Mach number approximation of $C$, while the line segment connecting $C'$ and $C''$ represents admissible generalized equilibria $E(\omega)$ [Eq. (16)] with $E(1)=C'$ and $E(\omega^*)=C''$ at $\omega^*=-1$.

defined as the solution of the minimization problem $f_G=\min_{f\in \mathbf{P}_G} H(f)$, where $\mathbf{P}_G\subset\mathbf{P}_M$ is the set of functions such that $\mathbf{P}_G=\{f>0:\langle f\rangle=\rho,\ \langle \boldsymbol{v}f\rangle=\rho\boldsymbol{u},\ \langle v_\alpha^2 f\rangle=\rho\Pi_{\alpha\alpha}\}$. In other words, minimization of the $H$ function Eq. (1) under the constraints of mass and momentum conservation and prescribed diagonal components of the pressure tensor yields

$$f_G=\rho\prod_{\alpha=x,y} w(v_\alpha)\frac{3(c^2-\Pi_{\alpha\alpha})}{2c^2}\left(\sqrt{\frac{\Pi_{\alpha\alpha}+cu_\alpha}{\Pi_{\alpha\alpha}-cu_\alpha}}\right)^{v_\alpha/c}\times\left(\frac{2\sqrt{\Pi_{\alpha\alpha}^2-c^2u_\alpha^2}}{c^2-\Pi_{\alpha\alpha}}\right)^{v_\alpha^2/c^2}. \quad (3)$$

To ease notation, we use $\Pi=(\Pi_{xx},\Pi_{yy})$ for a generic point on the two-dimensional plane of parameters. In order to ensure the positivity of $f_G$, it is required that $\Pi\in\Omega$ where $\Omega=\{\Pi: c|u_x|<\Pi_{xx}<c^2,\ c|u_y|<\Pi_{yy}<c^2\}$ is a convex rectangular in the plane of parameters for each velocity $\boldsymbol{u}$ (see Fig. 1).

The generalized Maxwellian (3) is the central result of this paper and is the key to the derivations below. It is interesting to note that while equilibrium (2) is analogous to the ordinary Maxwellian [spherically symmetric Gaussian $f_M\sim\exp\{-m(\boldsymbol{v}-\boldsymbol{u})^2/2k_B\Theta_0\}$, shifted from the origin by the amount of mean flow velocity $\boldsymbol{u}$ and with the width proportional to the fixed temperature $\Theta_0=c^2/3$], quasiequilibrium (3) resembles the anisotropic Gaussian, $f_G\sim\exp\{-(1/2)(\boldsymbol{v}-\boldsymbol{u})\cdot\Pi^{-1}\cdot(\boldsymbol{v}-\boldsymbol{u})\}$. The latter generalized Maxwellian corresponds to the ellipsoidal symmetry and is among the only few analytic results on the relevant distribution functions in the classical kinetic theory [15]. It is revealing that also in the LB realm the analog of the generalized Maxwellian has a nice closed form (3). The physical sense of Eq. (3) is that it distinguishes the relaxation of the diagonal components of the pressure tensor (and hence also of the trace of this tensor) among other nonconserved moments, and hence one expects a control over the dynamics of the trace which is responsible for the bulk viscosity (see below). Moreover, it is possible to evaluate explicitly the $H$ function in the generalized Maxwell states in Eq. (3), $H_G=H(f_G)$; the result is elegantly written as

$$H_G=\rho\ln\rho+\rho\sum_{\alpha=x,y}\sum_{k=-,0,+} w_k a_k(\Pi_{\alpha\alpha})\ln[a_k(\Pi_{\alpha\alpha})], \quad (4)$$

where $w_\pm=w(\pm1)$, $w_0=w(0)$, $a_\pm(\Pi_{\alpha\alpha})=3(\Pi_{\alpha\alpha}\pm cu_\alpha)/c^2$, and $a_0(\Pi_{\alpha\alpha})=3(c^2-\Pi_{\alpha\alpha})/(2c^2)$ (see Fig. 1).

Finally, with the help of $f_G$ [Eq. (3)], let us derive a constrained equilibrium $f_C$ which brings the $H$ function to a minimum among all the population lists with a fixed trace of the pressure tensor $T(\boldsymbol{\Pi})=\Pi_{xx}+\Pi_{yy}$. In terms of the parameter set $\Omega$, this is equivalent to require that the point $C=(\Pi_{xx}^C,\Pi_{yy}^C)$ belongs to a line segment $\mathbf{L}_T=\{\Pi\in\Omega:\Pi_{xx}+\Pi_{yy}=T\}$, and the constrained equilibrium $C$ is the one minimizing the function $H_G$ [Eq. (4)] on $\mathbf{L}_T$ (see Fig. 1). Since the restriction of a convex function to a line is also convex, the solution to the latter problem exists and is found by $[(\partial H_G/\partial\Pi_{xx})-(\partial H_G/\partial\Pi_{yy})]_{(\Pi_{xx},\Pi_{yy})\in\mathbf{L}_T}=0$, which yields a cubic equation in terms of the normal stress difference $N=\Pi_{xx}^C-\Pi_{yy}^C$,

$$N^3+aN^2+bN+d=0,$$

$$a=-\frac{1}{2}(u_x^2-u_y^2),\quad b=(2c^2-T)(T-u^2),$$

$$d=-\frac{1}{2}(u_x^2-u_y^2)(2c^2-T)^2. \quad (5)$$

Let us define $p=-a^2/3+b$, $q=2a^3/27-ab/3+d$, and $\Delta=(q/2)^2+(p/3)^3$. As long as $\Delta\geq0$, which is well satisfied in the low Mach number limit, the Cardano formula implies

$$\Pi_{xx}^C=\frac{T}{2}+\frac{1}{2}\left(r-\frac{p}{3r}-\frac{a}{3}\right),\quad r=\sqrt[3]{-\frac{q}{2}+\sqrt{\Delta}}, \quad (6)$$

while $\Pi_{yy}^C=T-\Pi_{xx}^C$. Thus, substituting Eq. (6) into Eq. (3), we find the constrained equilibrium

$$f_C=f_G[\rho,\boldsymbol{u},\Pi_{xx}^C(\boldsymbol{u},T),\Pi_{yy}^C(\boldsymbol{u},T)]. \quad (7)$$

Before proceeding any further, we mention that the generalized Maxwellian (3) is consistent with and extends the previously known results.

(i) The point of global minimum of the function $H_G$ [Eq. (4)] on $\Omega$ is found from $(\partial H_G/\partial\Pi_{\alpha\alpha})=0$. The corresponding solution $M=(\Pi_{xx}^M,\Pi_{xx}^M)$, where $\Pi_{\alpha\alpha}^M=-c^2/3+(2c^2/3)\sqrt{1+3(u_\alpha/c)^2}$, recovers the equilibrium $f_M$ [Eq. (2)] upon substitution into Eq. (3): $f_M=f_G[\rho,\boldsymbol{u},\Pi_{xx}^M(\boldsymbol{u}),\Pi_{yy}^M(\boldsymbol{u})]$.

(ii) In Ref. [7], a different LB equilibrium $f_\Theta$ was introduced as the entropy minimization problem under fixed density, momentum, and energy. That equilibrium was evaluated exactly only for vanishing velocity in [7] while a series expansion was used for $\boldsymbol{u}\neq0$. The present result reported above solves the problem of Ref. [7] *exactly* for any velocity. Substituting $T=2\Theta+u^2$ (two-dimensional ideal-gas equation of state with $\Theta$ as the temperature) into Eq. (7), it is simply

$f_\Theta(\rho,\boldsymbol{u},\Theta)=f_G[\rho,\boldsymbol{u},\Pi_{xx}^C(\boldsymbol{u},2\Theta+u^2),\Pi_{yy}^C(\boldsymbol{u},2\Theta+u^2)]$. Expanding the exact solution $\Pi_{xx}^C$ [Eq. (6)] in terms of the velocity $u$ yields the approximate solution consistent with Ref. [7], namely,

$$\Pi_{xx}^C = \Theta + \left(\frac{\Theta+1}{4\Theta}\right)u_x^2 + \left(\frac{3\Theta-1}{4\Theta}\right)u_y^2 + O(u^4). \tag{8}$$

(iii) In Ref. [16], a so-called guided equilibrium $\tilde{f}_\Theta$ was introduced in order to derive the LB method for compressible flows. That equilibrium is recovered by simply assuming the Maxwell-Boltzmann form of the diagonal components, $\Pi_{xx}=\Theta+u_x^2$ and $\Pi_{yy}=\Theta+u_y^2$, in Eq. (3): $\tilde{f}_\Theta(\rho,\boldsymbol{u},\Theta)=f_G(\rho,\boldsymbol{u},\Theta+u_x^2,\Theta+u_y^2)$.

Thus, the generalized Maxwellian (3) and its implication, the constrained equilibrium (7), unifies *all* the equilibria introduced previously on the D2Q9 lattice. Armed with the constrained equilibrium, we now proceed with the derivation of the kinetic equation. By means of the usual equilibrium $M$ and the presently found constrained equilibrium $C$, let us define the generalized equilibrium $E(\omega)=[\Pi_{xx}^E(\omega),\Pi_{yy}^E(\omega)]$ as a linear interpolation between the points $M$ and $C$ on the $\Pi$ plane,

$$E(\omega) = (1-\omega)M + \omega C, \tag{9}$$

where $\omega$ is a free parameter, and its admissible range will be defined next. Thus, the generalized equilibrium list is defined as

$$f_{\rm GE}(\omega) = f_G(\rho,\boldsymbol{u},\Pi_{xx}^E(\omega),\Pi_{yy}^E(\omega)). \tag{10}$$

Considering the kinetic equation of the form $\partial_t f+\boldsymbol{v}\partial_{\boldsymbol{x}} f=J(f)$, let us define the following collision operator:

$$J(f) = \lambda[f_{\rm GE}(\omega) - f], \tag{11}$$

where $\lambda>0$ is a parameter, ruling the relaxation toward the generalized equilibrium. In the continuum limit, $\lambda$ is related to the kinematic viscosity. While Eq. (11) reminds the popular Bhatnagar-Gross-Krook (BGK) model [17], the collision integral (11) depends on two parameters, $\lambda$ and $\omega$. In view of the analogy of $f_G$ with the anisotropic Gaussian, this is somewhat similar to the so-called ellipsoidal statistical model [17]. However, in our case, the leading order of the macroscopic equations recovered in the continuum limit does not depend on $\omega$, which is a tunable parameter for enhancing the stability of the corresponding LB scheme. Collision operator (11) conserves mass and momentum, i.e., $\langle J(f)\rangle=0$ and $\langle \boldsymbol{v}J(f)\rangle=0$. Note that at $\omega=0$, Eq. (11) reduces to the BGK LB model of Ref. [5], while at $\omega=1$ it becomes the so-called consistent LB model with energy conservation [7] (see remark (ii) above).

The key for proving the $H$ theorem for the kinetic equation is to establish the nonpositivity of the entropy production $\sigma$ due to the relaxation term (11), where

$$\sigma = \langle \ln(f/W)J(f)\rangle. \tag{12}$$

Clearly, if $f=f_M$, then $C=M$ and $\Pi^E(\omega)=\Pi^M$ for any $\omega$. From remark (i), it follows that entropy production annihilates at the equilibrium, $\sigma(f_M)=0$. In the general case, we have

$$\frac{\sigma}{\lambda} \le H_{\rm GE}(\omega) - H(f) \le H_{\rm GE}(\omega) - H_G(\Pi), \tag{13}$$

where $H_{\rm GE}(\omega)=H_G[\Pi_{xx}^E(\omega),\Pi_{yy}^E(\omega)]$. The first inequality is due to the convexity of the $H$ function, while the second holds because $f_G(\Pi_{xx},\Pi_{yy})$, by definition, minimizes $H$ among all the population lists with the moments $(\Pi_{xx},\Pi_{yy})$. Recalling that $\boldsymbol{\Pi}[f_{\rm GE}(1)]$ and $\boldsymbol{\Pi}[f_G(\Pi_{xx},\Pi_{yy})]$ have the same trace and taking into account the definition of the point $C$, inequality (13) can be rewritten as

$$\begin{aligned}\frac{\sigma}{\lambda} &\le H_{\rm GE}(\omega) - H_{\rm GE}(1) + H_{\rm GE}(1) - H_G(\Pi)\\ &\le H_{\rm GE}(\omega) - H_{\rm GE}(1).\end{aligned} \tag{14}$$

What remains is to estimate the range of $\omega$ such that $H_{\rm GE}(\omega)\le H_{\rm GE}(1)$. Clearly, since $M=E(0)$ is the absolute minimum of $H_G$ and because $H_{\rm GE}(\omega)$ is a convex function (a restriction of a convex function to a line), the right-hand side of Eq. (14) is nonpositive if $0\le\omega<1$. This proves nonpositivity of the entropy production in the interval $0\le\omega<1$. In order to extend the proof to $\omega<0$, let us consider the entropy estimate [5] (see also [18]),

$$H_{\rm GE}(\omega^*) = H_{\rm GE}(1). \tag{15}$$

Thanks to the convexity of $H_{\rm GE}(\omega)$, the nontrivial solution $\omega^*<0$ to this equation is unique when it exists. In the opposite case, we take $\omega^*<0$ from the condition, $E(\omega^*)\in\partial\Omega$, where $\partial\Omega$ is the boundary of the positivity domain $\Omega$. In both cases, for $\omega^*\le\omega\le0$, it holds $H_{\rm GE}(\omega)\le H_{\rm GE}(1)$. Thus, if $\omega$ takes values in the interval $\omega^*\le\omega<1$, the entropy production is nonpositive, $\sigma\le0$, which proves the existence of the $H$ theorem for the proposed model. Note that from the entropy estimate, it follows that $\omega^*$, in general, depends on the state $f$. However, Eq. (15) drastically simplifies at low Mach numbers which we consider next.

In the case of diffusion scaling [19,20], i.e., the flow regime with Kn$\sim$Ma$\sim u/c\ll1$, where Kn is the Knudsen number and Ma is the Mach number, Eq. (8) simplifies to $\Pi_{xx}^C=(T/2)+(\Pi_{xx}^M-\Pi_{yy}^M)/2+O(u^4)$ and similar to $\Pi_{yy}^C$. Introducing these results in Eq. (9) allows one to recast the definition of the generalized equilibrium, namely,

$$\Pi_{\alpha\alpha}^E(\omega) = \Pi_{\alpha\alpha}^M + \omega\frac{T-T_M}{2} + O(u^4). \tag{16}$$

Using Eq. (16) in the definition of the collision operator (11) and neglecting all the terms in the higher moments which are 2 order of magnitude (with regards to $u$) smaller than the corresponding terms required to recover incompressible Navier-Stokes equations [20], the collision operator can be simplified to

$$J'(f) = A(f_M - f), \tag{17}$$

where $A=\lambda B^{-1}\Lambda B$ and $9\times9$ matrices $B$ and $\Lambda$ are

$$\Lambda = \mathrm{diag}\left([0,0,0],\begin{bmatrix} r_+ & r_- \\ r_- & r_+\end{bmatrix},[1,1,1,1]\right),$$

$$B=[1,v_x,v_y,v_x^2,v_y^2,v_xv_y,v_x^2v_y,v_xv_y^2,v_x^2v_y^2]^T, \tag{18}$$

with $r_{\pm}=(r\pm 1)/2$ and $r=1-\omega$. Operator $J'$ is a MRT collisional operator with collision matrix $A$ (characterized by two relaxation frequencies $\lambda$ and $\delta=r\lambda$). It is possible to prove by means of the asymptotic analysis [21] that in the continuum limit, $J'$ leads to the kinematic (shear) viscosity and the second (bulk) viscosity coefficients given, respectively, by

$$\nu=\frac{c^2}{3\lambda}, \quad \xi=\frac{c^2}{3\delta}. \tag{19}$$

Finally, for low Mach numbers, the entropy $H_{\mathrm{GE}}$ can be estimated as follows:

$$H_{\mathrm{GE}}=\rho\ln\rho+\frac{3}{2}\rho u^2+\frac{9}{8}\rho(T-T_M)^2\omega^2+O(u^6). \tag{20}$$

Using Eq. (20) in the entropy condition (15), we find $\omega^* \approx -1$ (see Fig. 1). Consequently, the stability region of the relaxation frequency $\delta$ controlling the bulk viscosity is estimated $0<\delta<2\lambda$ or, taking into account Eq. (19), equivalently $0<\nu/\xi<2$. In particular, for high Reynolds number flows, the ratio $\nu/\xi$ tends to the lowest limit, i.e., large bulk viscosity is required to make the numerical computations more stable.

Since the bulk viscosity controls the attenuation of acoustic waves, which are fictitious when searching for the incompressible limit, increasing this tunable parameter allows one to mitigate the effects of fictitious compressibility and hence it increases the stability region of the scheme. In order to check the accuracy of the scheme, let us consider the Taylor-Green vortex flow test. Let us consider a square domain $(x,y)\in[0,2\pi/k]\times[0,2\pi/k]$. The Taylor-Green vortex flow has the following analytical solution [22]:

$$u_x=-u_0\cos(kx)\sin(ky)\exp(-2\nu k^2t), \tag{21}$$

$$u_y=u_0\cos(ky)\sin(kx)\exp(-2\nu k^2t), \tag{22}$$

$$p=-\frac{u_0^2}{4}[\cos(2kx)+\cos(2ky)]\exp(-4\nu k^2t), \tag{23}$$

where the pressure is $p=(c^2\rho)/(3\rho_0)$. It is immediate to prove that

$$\Phi(t)=\frac{1}{2}\int_0^{2\pi/k}\int_0^{2\pi/k}(u_x^2+u_y^2)k^2dxdy=\frac{u_0^2}{4}\exp(-4\nu k^2t). \tag{24}$$

The previous formula suggests a simple way to measure the actual kinematic viscosity. Introducing the simulation time $t\in[0,t_0]$, the measured kinematic viscosity is given by

$$\nu_*=-\frac{\ln[4\Phi(t_0)/u_0^2]}{4k^2t_0}. \tag{25}$$

In the following numerical results, we set $k=1$, $u_0=1$, $\rho_0=1$, and $t_0=5$. Consequently the Reynolds number becomes $\mathrm{Re}=2\pi/\nu$. Let us consider a homogeneous mesh made of $160\times160$ nodes, which implies a Knudsen number of $\mathrm{Kn}=1/160$. Let us select the Mach number as $\mathrm{Ma}=1/16$. Some numerical tests are reported for different kinematic viscosity $\nu$ and bulk viscosity $\xi$. The numerical results are reported in Table I and compared with the standard lattice BGK (LBGK) model. First of all, this test shows that the model recovers the right kinematic viscosity. Moreover, the relaxation frequency $\delta$, controlling the bulk viscosity, does not affect the leading part of the solution. According to the previous test, even large bulk viscosities may be adopted without affecting significantly the numerical results.

TABLE I. Taylor-Green vortex flow test. Some numerical tests are reported for different kinematic viscosity $\nu$ and bulk viscosity $\xi$. The mesh is made of $160\times160$ nodes. The Knudsen number is $\mathrm{Kn}=1/160$, the Mach number is $\mathrm{Ma}=1/16$, and finally the Reynolds number is $\mathrm{Re}=2\pi/\nu$. The actual kinematic viscosity $\nu_*$ is measured by means of Eq. (25) and the relative error $(\nu-\nu_*)/\nu$ is reported as well.

| | $\nu/\xi$ | $\nu$ | Measured $\nu_*$ | Error (%) |
|---|---|---|---|---|
| LBGK | 1 | 0.001 | 0.00102065 | 2.0648 |
| Present | 0.1 | 0.001 | 0.00102071 | 2.0713 |
| Present | 0.01 | 0.001 | 0.00102106 | 2.1058 |
| LBGK | 1 | 0.010 | 0.00998509 | −0.1491 |
| Present | 0.1 | 0.010 | 0.00998555 | −0.1445 |
| Present | 0.01 | 0.010 | 0.00998654 | −0.1346 |
| LBGK | 1 | 0.100 | 0.09977323 | −0.2268 |
| Present | 0.1 | 0.100 | 0.09977355 | −0.2264 |
| Present | 0.01 | 0.100 | 0.09977230 | −0.2277 |

To conclude, the generalized Maxwellian (3) opens a different perspective on the LB method. Various LB equilibria introduced in the past are special cases of Eq. (3). Important application of Eq. (3), considered in this paper, is a different class of entropic multiple relaxation-time (E-MRT) LB models. They enjoy both the $H$ theorem and the additional free-tunable parameter for controlling the bulk viscosity. Hence, they combine the two most successful strategies for enhancing stability of LB for high Reynolds number simulations. Because all the results above are derived in a closed form, numerical implementation of the E-MRT LB models is straightforward. Preliminary numerical results demonstrated that efficient stabilization of the LB simulation without loss of accuracy is indeed achieved with the suggested scheme. Moreover, the implementation is not much different from the familiar LBGK scheme, unlike the standard MRT model. These results show that the present model can be used for enhancing stability instead of the most popular LBGK method. Details of the implementation and numerical results will be reported in a separate publication.

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